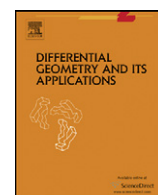


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## Surfaces in self-dual Einstein manifolds and their twistor lifts

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## ABSTRACT

In this note, we consider surfaces in self-dual Einstein manifolds whose twistor lifts are harmonic sections. In particular, we state the stability of the twistor lifts as harmonic sections and determine such surfaces of genus zero. This paper is a short survey of our previous results in Hasegawa (2007, 2009, 2011) [9–11].

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## 1. Introduction

In geometry of surfaces in oriented four-dimensional Riemannian manifolds, the twistor lifts, which are smooth maps from surfaces into the twistor spaces, play an important role and have been studied by many researchers (see [2,4,6,7], for example). In particular, surfaces are said to be superminimal if their twistor lifts are horizontal. The twistor space is endowed with a natural almost complex structure. It is well known that this almost complex structure is integrable if and only if a base manifold is self-dual (see [1]). Surfaces with holomorphic twistor lifts relative to the almost complex structure are also considered. Such surfaces are called twistor holomorphic surfaces. Note that a surface is superminimal if and only if it is minimal and twistor holomorphic.

On the other hand, for sections with unit length, we can consider the energy functional restricted to the space of all such sections and its stationary points, which are called harmonic sections (see [15], for example). Since the twistor lifts are sections of certain vector bundles with unit length, surfaces whose twistor lifts are harmonic sections can be considered. Note that the twistor lifts of twistor holomorphic surfaces are harmonic sections when the ambient spaces are self-dual Einstein. Recently surfaces whose twistor lifts are harmonic sections have been studied from the view point of the integrable systems in [3].

In this note, we report some results for such surfaces in self-dual Einstein manifolds which are proved in [9–11].

## 2. Sections of sphere bundles and harmonic sections

Throughout this paper, all manifolds and maps are assumed to be smooth. Let  $E$  be a vector bundle over a manifold  $M$  and  $E_x$  the fiber of  $E$  over  $x \in M$ . We write  $TP$  for the tangent bundle of a manifold  $P$ . For vector bundles  $E, E'$  over  $M$ , we denote the homomorphism bundle whose fiber is the space of linear mappings  $E_x$  to  $E'_x$  by  $\text{Hom}(E, E')$ , and set

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$\text{End}(E) := \text{Hom}(E, E)$ . The space of all sections of a vector bundle  $E$  is denoted by  $\Gamma(E)$ . Let  $\varphi : N \rightarrow M$  be a smooth map and  $E$  a vector bundle over  $M$ . The pull back bundle of  $E$  by  $\varphi$  is denoted by  $\varphi^*E$ .

In this section, we summarize the fundamental formulae for sections of the sphere bundles, which are obtained in [15]. Let  $E$  be a Riemannian vector bundle with a fiber metric  $g^E$  and a metric connection  $\nabla^E$  over an  $n$ -dimensional Riemannian manifold  $(M, g)$ . Let  $K^E : TE \rightarrow E$  be the connection map with respect to  $\nabla^E$ . The canonical metric  $G$  on  $E$  is defined by

$$G(\zeta, \zeta) = g(p_*(\zeta), p_*(\zeta)) + g^E(K^E(\zeta), K^E(\zeta))$$

for  $\zeta \in TE$ , where  $p : E \rightarrow M$  is the bundle projection. We note that  $p : (E, G) \rightarrow (M, g)$  is a Riemannian submersion with totally geodesic fibers. We call  $\ker p_*$  (resp.  $\ker K^E$ ) the vertical (resp. horizontal) subbundle of  $TE$ . For  $\xi \in \Gamma(E)$ , its vertical lift is denoted by  $\xi^v$ . For a vector field  $X$  on  $M$ ,  $X^h$  denotes the horizontal lift of  $X$ . We note that  $K^E(\xi^v) = \xi$  and  $K^E(\xi_*(X)) = \nabla_X^E \xi$  for  $\xi \in \Gamma(E)$  and  $X \in TM$ . We define  $H^{\nabla^E}$  by

$$H^{\nabla^E}(X, Y)\xi := -\nabla_X^E \nabla_Y^E \xi + \nabla_{\nabla_X Y}^E \xi$$

for  $X, Y \in \Gamma(TM)$  and  $\xi \in \Gamma(E)$ . The rough Laplacian  $\bar{\Delta}^{\nabla^E}$  of  $\nabla^E$  is defined by

$$\bar{\Delta}^{\nabla^E}(\xi) = \sum_{i=1}^n H^{\nabla^E}(e_i, e_i)(\xi) = -\sum_{i=1}^n (\nabla_{e_i}^E \nabla_{e_i}^E \xi - \nabla_{\nabla_{e_i} e_i}^E \xi)$$

for  $\xi \in \Gamma(E)$ , where  $e_1, \dots, e_n$  is an orthonormal frame of  $(M, g)$ . Set  $UE (= U(E)) := \{u \in E \mid g^E(u, u) = 1\}$  and consider the induced metric on  $UE$  from  $G$ . We assume that  $M$  is compact. Let  $\mathcal{E}$  be the energy functional defined on the space of all smooth maps from  $M$  to  $UE$ . For a section  $\xi \in \Gamma(UE)$ , the energy  $\mathcal{E}(\xi)$  is given by

$$\mathcal{E}(\xi) = \frac{n}{2} \text{Vol}((M, g)) + \frac{1}{2} \int_M \|\nabla^E \xi\|^2 dv, \quad (2.1)$$

where  $dv$  denotes the Riemannian measure of  $(M, g)$  and  $\text{Vol}((M, g))$  is the volume of  $(M, g)$ . We say that  $\xi \in \Gamma(UE)$  is a *harmonic section* if  $\xi$  is a stationary point of  $\mathcal{E}|_{\Gamma(UE)}$ . The second term of the right-hand side in (2.1) is called the *vertical energy* of  $\xi \in \Gamma(UE)$ . For a harmonic section  $\xi \in \Gamma(UE)$ , the Hessian at  $\xi$ , which is defined by the second variation formula, is denoted by  $\mathcal{H}_\xi$ . A variation vector field of  $\xi \in \Gamma(UE)$  can be identified with a section of  $E$  orthogonal to  $\xi$ . Let  $V_\xi$  be the set of all sections orthogonal to  $\xi \in \Gamma(UE)$ . The following fact is proved in [15].

**Lemma 2.1.** *Let  $\xi \in \Gamma(UE)$  be a harmonic section. Then, for  $\zeta_1, \zeta_2 \in V_\xi$ , the equation*

$$\mathcal{H}_\xi(\zeta_1, \zeta_2) = \int_M \{g^E(\nabla^E \zeta_1, \nabla^E \zeta_2) - \|\nabla^E \xi\|^2 g^E(\zeta_1, \zeta_2)\} dv$$

holds.

We say that a harmonic section  $\xi \in \Gamma(UE)$  is *weakly stable* if  $\mathcal{H}_\xi$  is positive semi-definite, that is,  $\mathcal{H}_\xi(\zeta, \zeta) \geq 0$  for all  $\zeta \in V_\xi$ . In particular, parallel sections with unit length are weakly stable harmonic sections.

### 3. Twistor spaces and twistor lifts for surfaces

Let  $(\tilde{M}, \tilde{g})$  be an oriented four-dimensional Riemannian manifold. The Hodge star operator is denoted by  $*$ . Since  $*^2 = \text{id}$  for all 2-forms, the bundle  $\Lambda^2(\tilde{M})$  of all 2-forms on  $\tilde{M}$  is decomposed into

$$\Lambda^2(\tilde{M}) = \Lambda_+^2(\tilde{M}) \oplus \Lambda_-^2(\tilde{M}),$$

where  $\Lambda_\pm^2(\tilde{M}) = \{\omega \in \Lambda^2(\tilde{M}) \mid *\omega = \pm\omega\}$ . Using the metrics, we can identify  $\Lambda_-^2(\tilde{M})$  with a vector subbundle  $Q$  of  $\text{End}(T\tilde{M})$ . We also write  $\tilde{g}$  for the fiber metric of  $Q$ . A section  $J \in \Gamma(UQ)$  satisfies  $J^2 = -I$ ,  $\tilde{g}(JX, JY) = \tilde{g}(X, Y)$  for all  $X, Y \in T\tilde{M}$  and  $-\Omega_J \wedge \Omega_J = d\mu$ , where  $\Omega_J$  is the fundamental form of  $J$  and  $d\mu$  is the volume form of  $\tilde{M}$  compatible with the orientation. Note that  $Q$  is a parallel subbundle in  $\text{End}(T\tilde{M})$  with respect to the connection which is induced by the Levi-Civita connection  $\tilde{\nabla}$  of  $\tilde{M}$ . We use the same letter  $\tilde{\nabla}$  for the induced connection. The twistor space  $\mathcal{Z}$  over  $\tilde{M}$  is the unit sphere bundle  $UQ$  of  $Q$ . The bundle projection  $p : \mathcal{Z} \rightarrow \tilde{M}$  and the connection  $\tilde{\nabla}$  induce the decomposition

$$T\mathcal{Z} = T^h\mathcal{Z} \oplus T^v\mathcal{Z}$$

into the horizontal subbundle  $T^h\mathcal{Z}$  and the vertical subbundle  $T^v\mathcal{Z}$ . On the twistor space  $\mathcal{Z}$ , a natural almost complex structure  $J^\mathcal{Z}$  is defined by  $J^\mathcal{Z}(X) = (J(p_*(X)))^h$  for all horizontal vectors  $X$  at  $J \in \mathcal{Z}$  and  $J^\mathcal{Z}(V) = J^v(V)$  for all vertical vectors  $V$ , where  $J^v$  is the canonical complex structure on each fiber ( $\simeq$  the two-dimensional unit sphere). Let  $\tilde{R}$  be

the curvature tensor of  $\tilde{g}$ . The curvature tensor  $\tilde{R}$  induces the endomorphism  $\tilde{\mathcal{R}}$  of  $\Lambda^2(\tilde{M})$ . It is well known that  $\tilde{\mathcal{R}}$  is decomposed into

$$\tilde{\mathcal{R}} = W^+ + W^- + N + \frac{\tau}{6} \text{id},$$

where  $W^+$  (resp.  $W^-$ ) is the self-dual (resp. anti-self-dual) part of the Weyl curvature tensor,  $N$  represents the traceless Ricci part and  $\tau$  is the scalar curvature. If  $W^- = 0$ , then  $\tilde{M}$  is called a *self-dual* manifold. The almost complex structure  $J^{\mathcal{Z}}$  on the twistor space is integrable if and only if  $W^- = 0$  (see [1]). If the Ricci tensor is proportional to the Riemannian metric, then  $\tilde{M}$  is called an *Einstein* manifold.

Let  $f : (M, g) \rightarrow (\tilde{M}, \tilde{g})$  be an isometric immersion from an oriented surface  $(M, g)$  into an oriented four-dimensional Riemannian manifold  $(\tilde{M}, \tilde{g})$ . The Levi-Civita connection of  $g$  is denoted by  $\nabla$ . Let  $T^\perp M$  be the normal bundle of  $f$  and  $\nabla^\perp$  the normal connection of  $T^\perp M$ . We say that an orthonormal frame  $e_1, e_2, e_3, e_4$  of  $f^*(T\tilde{M})$  is *adapted* if

- (1)  $e_1, e_2, e_3, e_4$  is compatible with the orientation of  $\tilde{M}$ ,
- (2)  $e_1, e_2$  is compatible with the orientation of  $M$ ,
- (3)  $e_3, e_4$  are normal to  $M$ .

Using an adapted frame, we define  $J : TM \rightarrow TM$  by  $J(e_1) = e_2$  and  $J(e_2) = -e_1$ , and  $J^\perp : T^\perp M \rightarrow T^\perp M$  by  $J^\perp(e_3) = -e_4$  and  $J^\perp(e_4) = e_3$ . It is evident that  $\nabla J = 0$  and  $\nabla^\perp J^\perp = 0$ . We set

$$\tilde{J}(X) := J(X) \quad \text{and} \quad \tilde{J}(\zeta) := J^\perp(\zeta)$$

for  $X \in TM$  and  $\zeta \in T^\perp M$ . Then  $\tilde{J}$  is a section of  $U(f^*Q) (= f^*(\mathcal{Z}))$  and  $\tilde{J}$  is called the *twistor lift* of  $M$ . Hereafter, we often omit the symbol “ $f$ ” for the induced objects of the immersion  $f$  if there is no confusion for the sake of simplicity. For example, we use the same letter  $\tilde{\nabla}$  for the pull back connection  $f^*\tilde{\nabla}$  by  $f$ . We say that  $M$  is a *superminimal* surface if the twistor lift is a horizontal map, that is,  $\tilde{\nabla}\tilde{J} = 0$ . If  $\tilde{J}_* \circ J = J^{\mathcal{Z}} \circ \tilde{J}_*$  (precisely,  $(f_\# \circ \tilde{J})_* \circ J = J^{\mathcal{Z}} \circ (f_\# \circ \tilde{J})_*$ , where  $f_\# : U(f^*Q) \rightarrow UQ$  is the bundle map), then  $M$  is called a *twistor holomorphic* surface. See [4] and [6] for twistor holomorphic surfaces, for example. Let  $\alpha$  (resp.  $H$ ) be the second fundamental form (resp. the mean curvature vector field) of  $M$ . We define a  $T^\perp M$ -valued symmetric tensor  $\beta$  by

$$\beta(X, Y) = \alpha(X, JY) - J^\perp \alpha(X, Y) + J^\perp \alpha(JX, Y) + \alpha(JX, Y)$$

for all  $X, Y \in TM$ . We see that  $M$  is twistor holomorphic if and only if  $\beta = 0$  (see [11]). Note that a surface is superminimal if and only if it is minimal and twistor holomorphic (see [6]). For  $\gamma \in \Gamma(\text{Hom}(TM \otimes TM, T^\perp M))$ , we define  $\nabla' \gamma$  by

$$(\nabla'_X \gamma)(Y, Z) = \nabla_X^\perp \gamma(Y, Z) - \gamma(\nabla_X Y, Z) - \gamma(Y, \nabla_X Z)$$

for all  $X, Y, Z \in \Gamma(TM)$ . A  $T^\perp M$ -valued 1-form  $\delta\gamma$  is defined by

$$(\delta\gamma)(X) = - \sum_{i=1}^2 (\nabla'_{u_i} \gamma)(u_i, X)$$

for all  $X \in TM$ , where  $u_1, u_2$  is an orthonormal frame of  $M$ . The following theorem holds (see [10]).

**Theorem 3.1.** *Let  $M$  be a surface in a self-dual Einstein manifold. Then the following statements are mutually equivalent:*

- (1) *The twistor lift  $\tilde{J}$  is a harmonic section.*
- (2) *The mean curvature vector  $H$  satisfies  $\nabla_{JX}^\perp H = J^\perp \nabla_X^\perp H$  for all  $X \in TM$ .*
- (3)  $\delta\beta = 0$ .

**Corollary 3.2.** *Let  $M$  be a twistor holomorphic surface in a self-dual Einstein manifold. Then the twistor lift of  $M$  is a harmonic section.*

From Theorem 3.1 or Corollary 3.2, we can see that there are many surfaces whose twistor lifts are harmonic sections (see also [10]). In particular, we note that the twistor lift of a Lagrangian surface  $M$  in a complex space form is a harmonic section if and only if  $M$  is Hamiltonian minimal with respect to a suitable orientation (see [3], for example). Here a Lagrangian immersion into a Kähler manifold is Hamiltonian minimal if it is a stationary point of the volume functional under all Hamiltonian variations. On the other hand, surfaces whose twistor lifts are harmonic sections are related to constrained Willmore surfaces. A conformal immersion of a Riemann surface is called a constrained Willmore surface if it is a stationary point of the Willmore functional under all conformal variations, which is a generalization of the Willmore surface. When the ambient space is  $\mathbf{R}^4$ , if the mean curvature vector field of  $M$  is a holomorphic section of  $T^\perp M$ , that is, the twistor lift of  $M$  is a harmonic section, then  $M$  is a constrained Willmore surface.

**Remark 1.** There exist surfaces whose twistor lifts are harmonic sections but not twistor holomorphic. In fact, the twistor lift of the product surface in  $\mathbf{R}^4$  of two circles in  $\mathbf{R}^2$  is a harmonic section, but this surface is not twistor holomorphic (see [9]).

**Remark 2.** Let  $G_o^{4,2}$  be the Grassmann manifold of all oriented 2-planes in  $\mathbf{R}^4$ . The Grassmann manifold  $G_o^{4,2}$  is isomorphic to  $S_+^2(1) \times S_-^2(1)$ , where  $S_\pm^2(1)$  are the unit spheres in the 3-dimensional vector spaces of all self-dual and anti-self-dual two-forms, respectively. Let  $p_\pm : G_o^{4,2} (\cong S_+^2(1) \times S_-^2(1)) \rightarrow S_\pm^2(1)$  be the projection onto each factor of the product of the spheres. On the other hand, the twistor space  $\mathcal{Z}$  of  $\mathbf{R}^4$  is  $\mathbf{R}^4 \times S^2(1)$ . The projection from  $\mathcal{Z}$  onto  $S^2(1)$  is denoted by  $\hat{p}$ . The map  $f_\# \circ \tilde{J}$  can be identified with the map  $M \ni x \mapsto \omega_1 \wedge \omega_2 - \omega_3 \wedge \omega_4$ , where  $\omega_1, \dots, \omega_4$  is the dual basis of an adapted basis  $e_1, \dots, e_4$  at each point  $x \in M$ . Then we have

$$(\hat{p} \circ f_\# \circ \tilde{J}) \circ \tilde{J} = p_- \circ \varphi,$$

where  $\varphi : M \rightarrow G_o^{4,2}$  is the Gauss map of  $M$  in  $\mathbf{R}^4$ . Therefore, if the twistor lift of  $M$  in  $\mathbf{R}^4$  is a harmonic section, then the half part of the Gauss map is harmonic. On the other hand, surfaces in Euclidean space whose (whole) Gauss map is harmonic have parallel mean curvature vector fields (see [14]).

#### 4. Stability of twistor lifts for twistor holomorphic surfaces in four-dimensional manifolds

Let  $f : (M, g) \rightarrow (\tilde{M}, \tilde{g})$  be an isometric immersion from an oriented surfaces  $(M, g)$  into an oriented four-dimensional Riemannian manifold  $(\tilde{M}, \tilde{g})$ . Let  $R^Q$  be the curvature form of the pull back connection on  $f^*Q$ . Obviously, the twistor lift of a superminimal surface is a harmonic section. In addition, if  $M$  is compact, it attains the minimum value  $\text{Vol}((M, g))$  for the restricted energy functional. Therefore, the twistor lifts of superminimal surfaces are weakly stable. In this section, we state the stability of the twistor lifts for surfaces in oriented four-dimensional Riemannian manifolds satisfying  $R_{X,Y}^Q \tilde{J} = 0$  for all  $X, Y \in TM$ . We denote the  $T^\perp M$ -component of  $\zeta \in T_{f(x)}\tilde{M}$  by  $\zeta^\perp$ .

**Lemma 4.1.** *Let  $(M, g)$  be an oriented surface in an oriented four-dimensional Riemannian manifold  $(\tilde{M}, \tilde{g})$ . Then the following statements are mutually equivalent:*

- (1)  $R_{X,Y}^Q \tilde{J} = 0$  for all  $X, Y \in TM$ .
- (2)  $(\tilde{R}_{X,Y} Z)^\perp + J^\perp(\tilde{R}_{X,Y} JZ)^\perp = 0$  for all  $X, Y, Z \in TM$ .
- (3)  $(\tilde{R}_{u,Ju} u)^\perp + J^\perp(\tilde{R}_{u,Ju} Ju)^\perp = 0$  for all  $u \in U(TM)$ .

Each condition above is related to the curvature operator as follows. We denote an adapted frame by  $e_1, e_2, e_3, e_4$  and its dual frame by  $\omega^1, \omega^2, \omega^3, \omega^4$ . Set

$$s_1 := \omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4, \quad s_2 := \omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^2, \quad s_3 := \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3.$$

Then,  $s_1, s_2, s_3$  is an orthonormal frame of  $f^*(\Lambda_-^2(\tilde{M}))$ . We obtain

$$\begin{aligned} & J^\perp((\tilde{R}_{e_1,Je_1})(Je_1))^\perp + ((\tilde{R}_{e_1,Je_1})(e_1))^\perp \\ &= -\hat{g}(\tilde{R}(\omega^1 \wedge \omega^2), \omega^1 \wedge \omega^3)e_3 - \hat{g}(\tilde{R}(\omega^1 \wedge \omega^2), \omega^1 \wedge \omega^4)e_4 \\ &\quad - \hat{g}(\tilde{R}(\omega^1 \wedge \omega^2), \omega^2 \wedge \omega^4)e_3 + \hat{g}(\tilde{R}(\omega^1 \wedge \omega^2), \omega^2 \wedge \omega^3)e_4 \\ &= -\hat{g}(\tilde{R}(\omega^1 \wedge \omega^2), s_2)e_3 - \hat{g}(\tilde{R}(\omega^1 \wedge \omega^2), s_3)e_4 \\ &= -\hat{g}(\omega^1 \wedge \omega^2, (W^- + N)s_2)e_3 - \hat{g}(\omega^1 \wedge \omega^2, (W^- + N)s_3)e_4. \end{aligned}$$

Using Lemma 4.1 and the equation above, we have

**Lemma 4.2.** *If  $\tilde{M}$  is self-dual Einstein manifold, then we have  $R_{X,Y}^Q \tilde{J} = 0$  for all  $X, Y \in TM$ .*

We set

$$\kappa := \tilde{g}(\tilde{R}(e_1, e_2)e_2, e_1) + \tilde{g}(\tilde{R}(e_1, e_2)e_3, e_4), \quad (4.1)$$

where  $e_1, e_2, e_3, e_4$  is an adapted frame. It is easy to see that  $\kappa$  does not depend on the choice of the frame  $e_1, e_2, e_3, e_4$ . We note that  $\tau = 12\kappa$  if  $\tilde{M}$  is a self-dual Einstein manifold with the scalar curvature  $\tau$ . For the stability of the twistor lifts as harmonic sections, we prove the following theorem in [10].

**Theorem 4.3.** Let  $M$  be an oriented surface in an oriented four-dimensional Riemannian manifold satisfying  $R_{X,Y}^Q \tilde{J} = 0$  for all  $X, Y \in TM$  and  $\kappa \geq 0$ . If  $M$  is twistor holomorphic, then the twistor lift  $\tilde{J}$  is a weakly stable harmonic section.

From Lemma 4.2 and Theorem 4.3, we have the following corollary.

**Corollary 4.4.** Let  $M$  be a twistor holomorphic surface in a self-dual Einstein manifold with nonnegative scalar curvature  $\tau$ . Then the twistor lift of  $M$  is a weakly stable harmonic section.

**Remark 3.** It is important to find non-parallel weakly stable harmonic sections of a Riemannian vector bundle  $E$ . Here we present examples of such sections:

- (1) The Hopf vector field on  $S^3(1)$  ( $E = TM$ ). See [12] and [15].
- (2) The unit normal vector field on a totally umbilic hypersurface  $\iota : S^n(r) \rightarrow S^{n+1}(1)$  with a suitable radius  $r$  and dimension  $n$  ( $E = \iota^*(TS^{n+1}(1))$ ). See [8].
- (3) The twistor lifts of certain non-superminimal compact twistor holomorphic surfaces in self-dual Einstein manifolds with nonnegative scalar curvature ( $E = f^*Q$ ). See Corollary 4.4.

Let  $\tilde{M}$  be a four-dimensional hyper-Kähler manifold and  $I_1, I_2, I_3$  a hyper-Kähler structure on  $\tilde{M}$ . If an orientation of  $\tilde{M}$  is given by

$$-\sum_{i=1}^3 \Omega_{I_i} \wedge \Omega_{I_i},$$

then we have  $I_1, I_2, I_3 \in \Gamma(\mathcal{Z})$ , where  $\Omega_{I_i}$  is the fundamental form of  $I_i$  ( $i = 1, 2, 3$ ). We note that  $\tilde{M}$  is a self-dual Einstein (in fact, Ricci flat) manifold with respect to this orientation. We denote the sectional curvature of  $\tilde{M}$  of the plane  $T_x \tilde{M}$  at each point  $x \in \tilde{M}$  by  $\tilde{K}(TM)$ . We have the following theorem in [10].

**Theorem 4.5.** Let  $\tilde{M}$  be a four-dimensional hyper-Kähler manifold and  $M$  an oriented, connected, compact surface in  $\tilde{M}$  such that

$$\int_M \tilde{K}(TM) dv \geq 0,$$

and there exist non-minimal points. If the twistor lift is a weakly stable harmonic section, then  $M$  is twistor holomorphic.

**Remark 4.** Let  $M$  be an oriented compact surface in a four-dimensional hyper-Kähler manifold. If  $\tilde{J}$  is a harmonic section, then  $\tilde{J}$  cannot be strongly stable, that is, the Hessian  $\mathcal{H}_{\tilde{J}}$  is not positive definite. We define the functions  $a_i$  on  $M$  by  $a_i = \tilde{g}(I_i, \tilde{J})$  for  $i = 1, 2, 3$ . Set  $\bar{I}_i := I_i - a_i \tilde{J}$  ( $i = 1, 2, 3$ ). Since  $\tilde{J}$  is a harmonic section, we have  $\Delta a_i = \|\tilde{\nabla} \tilde{J}\|^2 a_i$  ( $i = 1, 2, 3$ ), where  $\Delta$  is the Laplacian acting on functions on  $M$ . From Lemma 2.1 and the Green's formula, we obtain

$$\mathcal{H}_{\tilde{J}}(\bar{I}_i, \bar{I}_i) = \int_M (3(\Delta a_i) a_i - \|\tilde{\nabla} \tilde{J}\|^2) dv = \int_M (3a_i^2 - 1) \|\tilde{\nabla} \tilde{J}\|^2 dv$$

for all  $i$ . Therefore, it holds that

$$\sum_{i=1}^3 \mathcal{H}_{\tilde{J}}(\bar{I}_i, \bar{I}_i) = 0.$$

Then,  $\tilde{J}$  cannot be strongly stable.

When  $\tilde{M} = \mathbf{R}^4$ , surfaces whose twistor lifts are weakly stable harmonic sections are completely determined by Theorem 4.5 as follows.

**Corollary 4.6.** Let  $M$  be an oriented, connected, compact surface in  $\mathbf{R}^4$ . If the twistor lift is a weakly stable harmonic section, then  $M$  is twistor holomorphic.

## 5. Surfaces of genus zero whose twistor lift are harmonic sections

In this section, we study surfaces of genus zero in self-dual Einstein manifolds whose twistor lift are harmonic sections. Let  $\chi(T^\perp M)$  be the Euler characteristic of  $T^\perp M$ . We obtain the following theorem in [11].

**Theorem 5.1.** *Let  $\tilde{M}$  be a self-dual Einstein manifold with the scalar curvature  $\tau$  and  $M$  an oriented, connected, compact surface in  $\tilde{M}$ . If the twistor lift of  $M$  is a harmonic section and the genus of  $M$  is zero, then we have*

- (1)  $M$  is a non-superminimal minimal surface when  $\chi(T^\perp M) > 2 - (\tau/24\pi) \text{Vol}(M)$ ,
- (2)  $M$  is a superminimal surface when  $\chi(T^\perp M) = 2 - (\tau/24\pi) \text{Vol}(M)$ ,
- (3)  $M$  is a non-superminimal twistor holomorphic surface when  $\chi(T^\perp M) < 2 - (\tau/24\pi) \text{Vol}(M)$ .

We note that  $\chi(T^\perp M)$  is an even integer if  $\tilde{M}$  is a hyper-Kähler manifold. From Theorem 5.1, we have the following corollary.

**Corollary 5.2.** *Let  $\tilde{M}$  be a four-dimensional hyper-Kähler manifold and  $M$  an oriented, connected, compact surface in  $\tilde{M}$ . If the twistor lift of  $M$  is a harmonic section and the genus of  $M$  is zero, then we have*

- (1)  $M$  is a non-superminimal minimal surface when  $\chi(T^\perp M) \geq 4$ ,
- (2)  $M$  is a superminimal surface when  $\chi(T^\perp M) = 2$ ,
- (3)  $M$  is a non-superminimal twistor holomorphic surface when  $\chi(T^\perp M) \leq 0$ .

If  $M$  is a non-superminimal twistor holomorphic surface in a hyper-Kähler manifold, then  $\chi(T^\perp M) \leq 0$  [9]. In the case where  $\tilde{M} = \mathbf{R}^4$ , we can obtain the following corollary immediately.

**Corollary 5.3.** *Let  $M$  be an oriented, compact, connected surface of genus zero in  $\mathbf{R}^4$ . If the twistor lift of  $M$  is a harmonic section, then  $M$  is a non-superminimal twistor holomorphic surface.*

Since the property that  $M$  is twistor holomorphic is invariant under conformal changes of the metric on  $\tilde{M}$ , we can obtain many twistor holomorphic surfaces of genus zero in  $\mathbf{R}^4$  from superminimal surfaces in the four-dimensional sphere. In general, the holomorphic mean curvature vector field is not parallel. In the case where the mean curvature vector is parallel, using Theorem 5.1, we have the following corollary (see [5] and [13], for example).

**Corollary 5.4.** *Let  $M$  be an oriented, connected, compact surface of genus zero in the space form of constant curvature. If the mean curvature vector field is parallel with respect to the normal connection, then  $M$  is minimal or totally umbilic.*

It is easy to obtain the Hopf's theorem for a constant mean curvature surface of genus zero in  $\mathbf{R}^3$  using by Corollary 5.4. In fact, by considering the totally geodesic immersion from  $\mathbf{R}^3$  to  $\mathbf{R}^4$ , the constant mean curvature surface can be seen as immersed surface in  $\mathbf{R}^4$  with parallel mean curvature vector field. Thus our main theorem is a generalization of the Hopf's theorem.

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